The Physics of Racing, Part 26: The Driving Wheel, Chapter I

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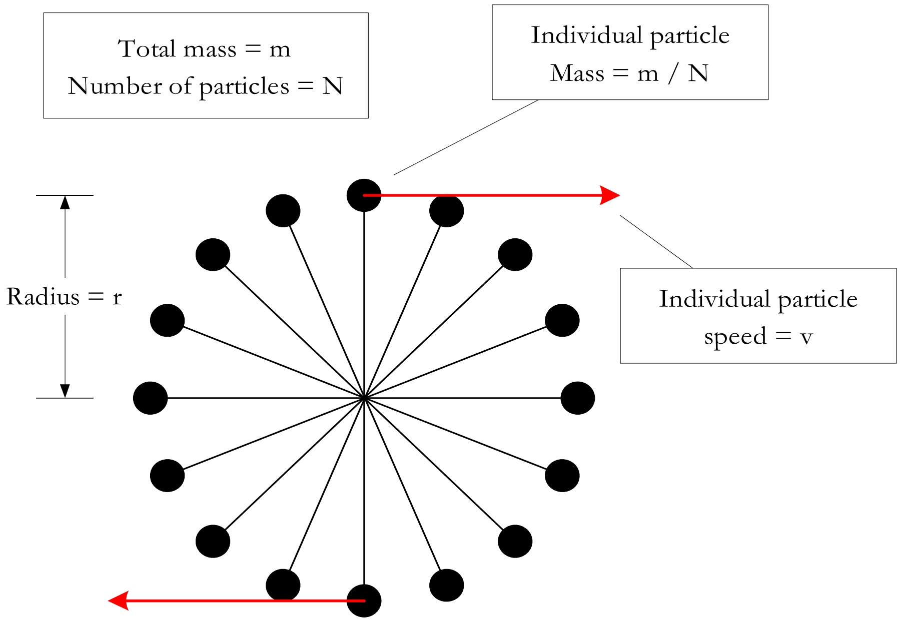
Imagine 400 ft-lbs of torque measured on a chassis dynamometer like a DynoJet (see references at the end). This is a very nice number to have in any car, street or racing. In dyno-speak, however, the interpretation of this number is a little tricky.

To start a dyno session, you strap your car down with the driving wheels over a big, heavy drum, then you run up the gears gently and smoothly until you're in fourth at the lowest usable engine RPM, then you floor it, let the engine run all the way to redline, then shut down. The dyno continuously measures the time and speed of the drum and the engine RPM through a little remote radio receiver that picks up spark-plug noise. The only things resisting the motion of the driving wheels are the inertia of the driveline in the car and the inertia of the drum. The dyno 'knows' the latter, but not the former. Without these inertias loading the engine, it would run up very quickly and probably blow up. Test-stand dynos, which run engines out of the car, load them in different ways to prevent them from free running to annihilation. Some systems use water resistance, others use electromagnetic; in any case, the resistance must be easy to calibrate and measure. We are only concerned with chassis dynos in this article, however.

What is the equation of motion for the car + dyno system? It is a simple variation on the theme of the old, familiar second law of Newton. For linear motion, that law has the form $F = ma$, where F is the net force on an object, m is its mass, and a is its acceleration, or time rate of change of velocity.

For rotational motion, like that of the driveline and dyno, Newton's second law takes the form $T = J\dot{\omega}$ where T is the net torque on an object, J is its ***moment of inertia***, and $\dot{\omega}$ is its ***angular acceleration***, or time rate of change of ***angular velocity***. The purpose of this instalment of the Physics of Racing is to explain everything here and to run a few numbers.

To get the rotational equation of motion, we assume that the dyno drum is strong enough that it will never fly apart, no matter how fast it spins. We model it, therefore, as a bunch of point masses held to the centre of rotation by infinitely strong, massless cables. With enough point masses, we can approximate the smooth (but grippy) surface of the dyno drum as closely as we would like.



Assume each particle receives a force of F/N in the tangential direction. Tangential, of course, means the same as circumferential or longitudinal, as clarified in recent instalments about slip and grip of tires. So, each particle accelerates according to $F/N = ma/N$. The N cancels out, leaving $a = F/m$. Now, a is the rate of change of the velocity, and the velocity is defined as $v = r\omega$, where r is the constant radius of the circle and ω is the angular velocity in ***radians*** per second. The circumference of the circle is $2\pi r$, by definition, so a drum of 3-foot radius has a circumference of about 6.28x3=18.8 feet. At 60 RPM, which is one rev per second, each particle goes 18.8 feet per second, which is about 15x18.8/22=12.8 mph. RPM is one measurement of angular velocity, but it's more convenient to measure it such that 2π angular units go by every second. Such units save us from having to track factors of 2π all over the math. So, there are 2π radians per revolution, and the equation $v = r\omega$ is seen as a general expression of the example that $2\pi \times$ revolutions per second = 18.8 feet per second = velocity.

Since r is constant, it has no rate of change. Only ω has one, measured in radians per second per second, or radians per second squared, or rad/sec^2, and denoted with an overdot: $\dot{\omega}$. The equation of motion, so far, looks like $a = \dot{v} = \dot{\omega}r = F/m$. Now, we know that torque is just force times the lever arm over which the force is applied. So, a force of F at the surface of the drum translates into a torque of $T = Fr$ applied to the shaft—or by the shaft, depending on point of view. So we write $\dot{\omega}r = F/m = T/rm$, which we can rearrange to $\dot{\omega}mr^2 = T$. We can make this *resemble* the linear form of Newton's law if we *define* $J = mr^2$, the moment of inertia of the drum, yielding $T = J\dot{\omega}$, which looks just like $F = ma$ if we analogize as follows: $F \leftrightarrow T$, $J \leftrightarrow m$, $a = \dot{v} \leftrightarrow \dot{\omega}$.

This value for J only works for this *particular* model of the drum, with all the mass elements at distance r from the center. Suffice it to say that a moment of inertia for any other model of the drum could be computed in like manner. It turns out that the moment of inertia of a solid cylindrical drum is half as much, namely $J = \frac{1}{2}mr^2$.

Moments of inertia for common shapes can be looked up all over the place, for instance at http://www.physics.uoguelph.ca/tutorials/torque/Q.torque.inertia.html.

So, now, the dyno has a known, fixed value for J, and it measures $\dot{\omega}$ very accurately. This enables it to calculate trivially just how much torque is being applied by the driving wheels of the car to the drum. But it does *not* know the moment of inertia of the driveline of the car, let alone the radius of the wheels, the gear selected by the driver, the final-drive ratio in the differential, and so on. In other words, it knows nothing about the driveline other than engine RPM.

Everyone knows that the transmission and final-drive on a car *multiply* the engine torque. The torque at the driving wheels is almost always much larger than the flywheel torque, and it's larger in lower gears than in higher gears. So, if you run up the dyno in third gear, it will accelerate faster than if you run it up in fourth gear. Yet, the dyno reports will be comparable. Somehow, without knowing any details about the car, not even drastic things like gear choice, the dyno can figure out flywheel torque. Well, yes and no.

It turns out that all the dyno needs to know is engine RPM. It does not matter whether the dyno is run up quickly with a relatively large drive-wheel torque (***DWT***) or run up slowly with a relatively small DWT. Furthermore, the radius of the driving wheels and tires also does not matter. Here's why.

Wheel RPM is directly proportional to drum RPM, assuming the longitudinal slip of the tires is within a small range. The reason is that at the point of contact, the drum and wheel have the same circumferential (longitudinal, tangential) speed, so $v_{\text{wheel}} = r_{\text{wheel}}\omega_{\text{wheel}} = v_{\text{drum}} = r_{\text{drum}}\omega_{\text{drum}}$. Let's write $\omega_{\text{wheel}} = A\omega_{\text{drum}}$, where $A = r_{\text{drum}}/r_{\text{wheel}}$. Engine RPM is related to wheel RPM by a factor that depends on the final-drive gear ratio f and the selected gear ratio $g_i, i = 1,2,3,\ldots$. We write $\omega_{\text{engine}} = B(f, g_i)\omega_{\text{wheel}}$. Usually, engine RPM is much larger than wheel RPM, so we can expect $B(f, g_i)$ to be larger than 1 most of the time. So, we get

$$\omega_{\text{drum}} = \frac{\omega_{\text{wheel}}}{A} = \frac{\omega_{\text{engine}}}{AB(f, g_i)}$$

We also know that, by Newton's Third Law, that the force applied to the drum by the tire is the same as the force applied to the tire by the drum. Therefore the torques applied are in proportion to the radii of the wheel+tyre and the drum, namely that

$$F_{\text{wheel}} = \frac{T_{\text{wheel}}}{r_{\text{wheel}}} = F_{\text{drum}} = \frac{T_{\text{drum}}}{r_{\text{drum}}}$$

or $T_{\text{drum}} = AT_{\text{wheel}}$. Recalling that the transmission gear and final drive multiply engine torque, we also know that $T_{\text{wheel}} = B(f, g_i)T_{\text{engine}}$, so $T_{\text{drum}} = AB(f, g_i)T_{\text{engine}}$. But we already know $AB(f, g_i)$: it's the ratio of the RPMs, so $T_{\text{drum}} = \frac{\omega_{\text{engine}}}{\omega_{\text{drum}}}T_{\text{engine}}$, or, more usefully,

$$T_{\text{engine}} = \frac{\omega_{\text{drum}}}{\omega_{\text{engine}}} T_{\text{drum}} = \frac{\omega_{\text{drum}}}{\omega_{\text{engine}}} J \dot{\omega}_{\text{drum}}$$

Every term on the right-hand side of this equation is measured or known by the dyno, so we can measure engine torque independently of car details! We can even plot T_{engine} versus ω_{engine}, effectively taking the run-up time and the drum data out of the report.

Almost. There is a small gotcha. The engine applies torque indirectly to the drum, spinning it up. But the engine is *also* spinning up the clutch, transmission, drive shaft, differential, axles, and wheels, which, all together, have an unknown moment of inertia that varies from car-to-car, though it's usually considerably smaller than J, the moment of inertia of the drum. But, in the equations of motion, above, we have not accounted for them. More properly, we should write

$$T_{\text{engine}} = \frac{\omega_{\text{drum}}}{\omega_{\text{engine}}} \left(J + J_{\text{miscellaneous}} \right) \dot{\omega}_{\text{drum}}$$

This doesn't help us much because we don't know $J_{\text{miscellaneous}}$, so we pull a fast one and rearrange the equation:

$$\text{define } T_{\text{loss}} = \frac{\omega_{\text{drum}}}{\omega_{\text{engine}}} J_{\text{miscellaneous}} \dot{\omega}_{\text{drum}}$$

$$T_{\text{engine}} - T_{\text{loss}} = \frac{\omega_{\text{drum}}}{\omega_{\text{engine}}} J \dot{\omega}_{\text{drum}}$$

This is why chassis dyno numbers are always lower than test-stand dyno numbers for the same engine. The chassis dyno measures $T_{\text{engine}} - T_{\text{loss}}$, and the test-stand measures T_{engine}. Of course, those trying to sell engines often report the best-sounding numbers: the test-stand numbers. So, don't be disappointed when you take your hot, new engine to the chassis dyno after installation and get numbers 15% to 20% lower than the advertised 'at the crankshaft' numbers in the brochure. It's to be expected. Typically, however, you simply do not know T_{loss}: it's a number you take on faith.

Let's run a quick sample. The following numbers are pulled out of thin air, so don't hang me on them. Suppose the drum has 3-foot radius, is solid, and weighs 6,400 lbs, which is about 200 slugs (remember slugs? One slug of mass weighs about 32 pounds at the Earth's surface). So, the moment of inertia of the drum is about $\frac{1}{2} m r^2 = 900 \text{ slug-ft}^2$. Let's say that the engine takes about 15 seconds to run from 1,500 RPM to 6,000 RPM in fourth gear, with a time profile like the following:

t	e RPM	V MPH	v FPS	drum ω	drum $\dot{\omega}$	drum RPM	RPM ratio	Torque
0	1,500	35	51.33	17.11	0.00	163.40	0.1089	0.00
1	1,800	42	61.60	20.53	3.42	196.08	0.1089	335.51
2	2,100	49	71.87	23.96	3.42	228.76	0.1089	335.51
3	2,400	56	82.13	27.38	3.42	261.44	0.1089	335.51
4	2,700	63	92.40	30.80	3.42	294.12	0.1089	335.51
5	3,000	70	102.67	34.22	3.42	326.80	0.1089	335.51
6	3,300	77	112.93	37.64	3.42	359.48	0.1089	335.51
7	3,600	84	123.20	41.07	3.42	392.16	0.1089	335.51
8	3,900	91	133.47	44.49	3.42	424.84	0.1089	335.51
9	4,200	98	143.73	47.91	3.42	457.52	0.1089	335.51
10	4,500	105	154.00	51.33	3.42	490.20	0.1089	335.51
11	4,800	112	164.27	54.76	3.42	522.88	0.1089	335.51
12	5,100	119	174.53	58.18	3.42	555.56	0.1089	335.51
13	5,400	126	184.80	61.60	3.42	588.24	0.1089	335.51
14	5,700	133	195.07	65.02	3.42	620.92	0.1089	335.51
15	6,000	140	205.33	68.44	3.42	653.60	0.1089	335.51

The "v MPH" column is just a straight linear ramp from 35 MPH to 140 MPH, which are approximately right in my Corvette. The "v FPS" column is just 22/15 the v MPH. The drum ω is in radians per second and is just v FPS divided by 3 ft, the drum radius. The drum $\dot{\omega}$ is just the stepwise difference of the drum ω numbers. It's constant, as we would expect from a run-up of the dyno at constant acceleration. The drum RPM is $60/2\pi$ times the drum ω. The RPM ratio is just drum RPM divided by engine RPM, and it must be strictly constant, so this is a nice sanity check on our math. Finally, the torque column is the RPM ratio times $J = 900$ slug – ft^2 times drum $\dot{\omega}$. We see a constant torque output of about 335 ft-lbs. Not bad. It implies a test-stand number of between 394 and 418, corresponding to 15% and 20% driveline loss, respectively. Looks like we nailed it without 'cooking the books' too badly. Of course, we have a totally flat torque curve in this little sample, but that's only because we have a completely smooth ramp-up of velocity.

Dyno reports often will be labelled 'Rear-wheel torque' (RWT) or, less prejudicially, 'drive-wheel torque' (DWT) to remind the user that there is an unknown component to the measurement. These are well-intentioned *misnomers*: do not be mislead! What they mean is 'engine torque as if the engine were connected to the drive wheels by a massless driveline', or 'engine torque as measured at the drive wheels with an unknown but relatively small inertial loss component'. It should be clear from the above that the actual drive-wheel torque cannot be measured without knowing A, the ratio of the drum radius to the wheel+tyre radius. It's slightly ironic that an attempt to clear up the confusion risks introducing more confusion.

In the next instalment, we relate the equations of motion for the driving wheel to the longitudinal magic formula to compute reaction forces and get equations of motion for the whole car.

References:

http://www.c5-corvette.com/DynoJet_Theroy.htm [sic]

http://www.mustangdyne.com/pdfs/7K%20manualv238.pdf

http://www.revsearch.com/dynamometer/torque_vs_horsepower.html

Attachments:

I've included the little spreadsheet I used to simulate the dyno run. It can be downloaded here.

ERRATA:

* Part 14, yet again, the numbers for frequency are actually in radians per second, not in cycles per second. There are 2π cycles per radian, so the 4 Hz natural suspension frequency I calculated and then tried to rationalize was really 4/6.28 Hz, which is quite reasonable and not requiring any rationalization. Oh, what tangled webs we weave…

* Physical interpretations of slip on page 2 of part 24: "Car (hub) moving forward, CP moving slowly forward w.r.t. ground, resisting car motion." Should be "Car (hub) moving forward, CP moving slowly forward w.r.t. ***HUB***, resisting car motion."

* Part 21, in the back-of-the-envelope numerical calculation just before the 3-D plot at the end of the paper, I correctly calculated $\tan^{-1}(0.822) = 0.688$, but then incorrectly calculated $\tan^{-1}(SB) - SB$ as -0.266. Of course, it's $0.688 - 0.822 = -0.134$. One of the hazards of doing math in one's head all the time is the occasional slip up. Normally, I check results with a calculator just to be really sure, but some are so trivial it just seems unnecessary. Naturally, those are the ones that bite me.