The Physics of Racing, Part 25: Combination Grip

Brian Beckman, PhD

In this instalment of the *Physics of Racing*, we complete the program begun last time to combine the magic formulae of parts 21 and 22, so that we have a model of tyre forces when turning and braking or turning and accelerating at the same time. Parts 21 and 22 introduced the magic formulae. The first one takes longitudinal slip as input and produces longitudinal grip as output. The other one takes lateral slip as input and produces lateral grip. Slip depends primarily on driver inputs, grip is force generated at the ground. *Longitudinal* means in the straight-ahead direction. *Lateral* means sideways, as in the forces for turning. Since the magic formulae work only in isolation, we have work to do to model turning and braking at the same time and turning and accelerating at the same time.

Last time, we vectorised slip—the input—to come up with ***combination slip***, captured in the vector ***slip velocity***. That vector measures the velocity of the contact patch with respect to (w.r.t.) the ground in one, handy definition. This time, we first boil down combination slip to new inputs for the old magic formulae. In the old magic formulae, we measure longitudinal slip as a percentage of unity, that is, as a percentage of breakaway sliding; and we measure lateral slip as an angle in degrees. These are not ***commensurable***, meaning that we do not use the same units of measurement for both kinds of slip. That's why there was a big, fat question mark in the vector slot for combination slip in one of the tables in part 24. Once we make them commensurable, then we stitch the magic formulae together to get one vector gripping force as a function of one vector slip. This finally allows us to compute the forces delivered by a tyre under combination control inputs.

Once again, we are in uncharted territory, so take it all in the for-fun spirit of this whole series of articles. I don't represent anything I do here as authoritative racing practice. I only claim to be bringing the fresh perspective of a stubbornly naïve physicist to the problems of racing cars as an amateur. The standard practice of the professional racing engineering community may be completely different. This is the *Physics of Racing*, not the *Engineering of Racing*. I'm after the fundamental principles behind the game. I use techniques that may be foreign to the engineers that build and race cars professionally. My results may not be precise enough for final application. I may take approximations that simplify away things that are actually critically important. On purpose, I'm figuring things out on my own. Often, this helps me understand published engineering information better. Just as often, it helps me debunk and debug the conventional wisdom. If you find mistakes, gaffs, or laughable dumb stuff, or if you know better ways to do things, I encourage you to fire up debate, publish rebuttals, or write to me directly. I've done my best to track down the latest and greatest information, but I've found lots of errors, ambiguities, and inexplicabilities in the open literature. I also suspect a conspiracy, meaning that I'd bet that the tyre manufacturers and pro racing teams don't publish their best information—I certainly wouldn't if I were they.

Disclaimers out of the way, we now have enough tools on the table to combine the two magic formulae. Recall the formulae from parts 21 and 22: $F_x(\sigma, F_z)$ and $F_y(\alpha, F_z, \gamma)$ for the longitudinal and lateral forces. Here they are, in isolation:

$$\boxed{\begin{aligned}
&F_x(\sigma, F_z) = D \sin\left(b_0 \tan^{-1}\left\{SB + E\left[\tan^{-1}(SB) - SB\right]\right\}\right) + S_u \\
&\qquad D = \mu_{xp} F_z = (b_1 F_z + b_2) F_z, \quad B = (b_3 F_z + b_4)\exp(-b_5 F_z) / b_0 (b_1 F_z + b_2) \\
&\qquad E = \left(b_6 F_z^2 + b_7 F_z + b_8\right), \qquad S = (100\,\sigma + b_9 F_z + b_{10}), \quad S_u = \text{constant}
\end{aligned}}$$

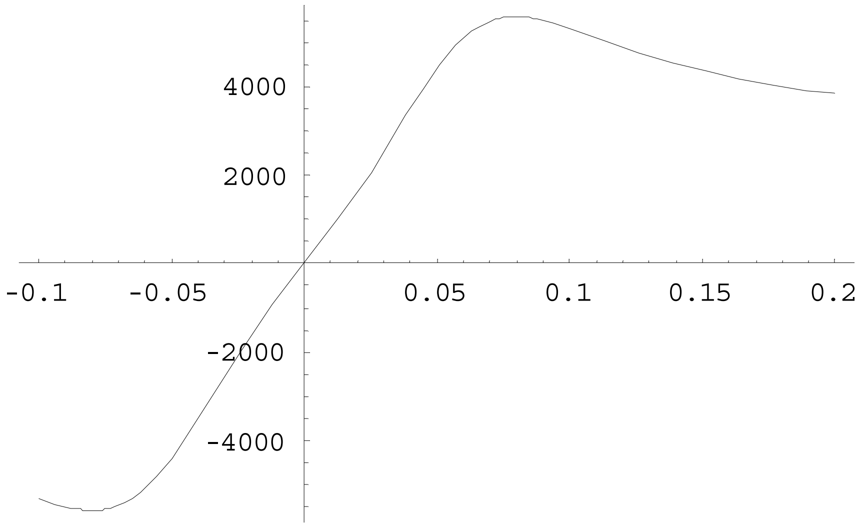
$$\boxed{\begin{aligned}
&F_y(\alpha, F_z, \gamma) = D \sin\left(a_0 \tan^{-1}\left\{SB + E\left[\tan^{-1}(SB) - SB\right]\right\}\right) + S_v \\
&\qquad D = \mu_{yp} F_z = (a_1 F_z + a_2) F_z, \quad B = a_3 \sin\left[2\tan^{-1}(F_z / a_4)\right](1 - a_5 |\gamma|) / a_0 (a_1 F_z + a_2) F_z \\
&\qquad E = a_6 F_z + a_7, \qquad\qquad S = \alpha_{\text{degrees}} + a_8 \gamma_{\text{degrees}} + a_9 F_z + a_{10} \\
&\qquad S_v = \left[(a_{11,1} F_z + a_{11,2})\gamma_{\text{degrees}} + a_{12}\right] F_z + a_{13}
\end{aligned}}$$

There are a lot of ways we could stitch them together. This is not the kind of situation where there is one right answer. Instead, in the absence of hard theory or experimental data, we have the freedom to be creative, with the inevitable risk of being wrong. We pick a method that satisfies some simple, intuitive, physical requirements. First, we must put the inputs on the same footing. Ask "what is the value of σ for which $F_x(\sigma, F_z)$ has its maximum, and what is the value of α for which $F_y(\alpha, F_z, \gamma)$ has its maximum?" Call these two values $\bar{\sigma}$ and $\bar{\alpha}$. They are constants for given F_z and γ: characteristics of a particular tyre and car and surface. So, we can finesse the notation and just write $F_x(\sigma)$ and $F_y(\alpha)$. The maxima identify points on the rim or edge of the 'traction circle'. The grip decreases when σ exceeds $\bar{\sigma}$ and when α exceeds $\bar{\alpha}$. Let's illustrate with $F_z = 3.3\text{KN}$, $\gamma = 0$, and the constants from Genta's alleged Ferrari. Once we substitute all that in (and we'll let you check our arithmetic from the data in prior articles), we get

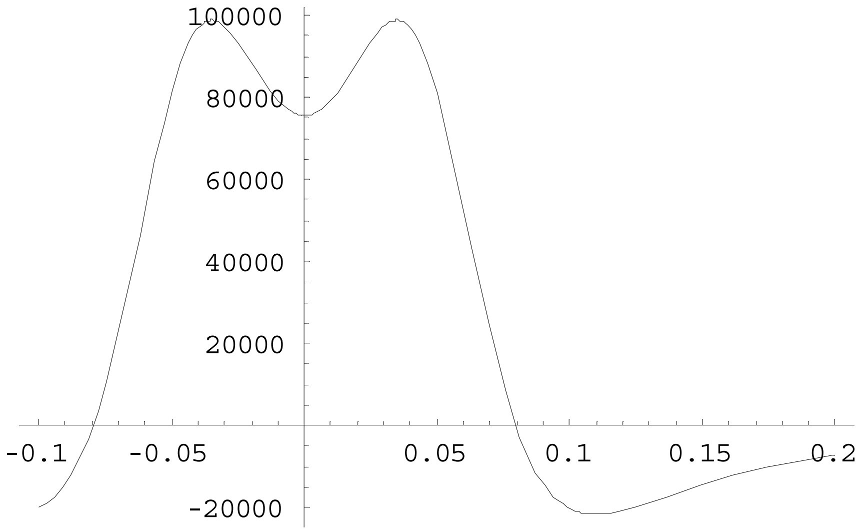
$$\begin{aligned}
&F_{x,\,\text{Newtons}}(\sigma) = 5570 \sin\left(1.65 \tan^{-1}\left\{SB - 10\left[\tan^{-1}(SB) - SB\right]\right\}\right) \\
&\qquad SB = 8.222\,\sigma
\end{aligned}$$

$$\begin{aligned}
&F_{y,\,\text{Newtons}}(\alpha) = 5570 \sin\left(1.799 \tan^{-1}\left\{SB + E\left[\tan^{-1}(SB) - SB\right]\right\}\right) \\
&\qquad B = 0.348, \quad E = -0.184, \quad S = \left[\alpha_{\text{degrees}} - 0.0524\right]
\end{aligned}$$

We evaluate these equations for $\sigma = 0$, $\alpha = 0$, getting $F_x(0) = 0$, $F_y(0) = -71\,\mathrm{N}$, and showing a small lateral force (about 16 lbs) due to conicity and ply steer. The source of that problem is the constant offset in *S*, which results from a_9 and a_{10}'s being non-zero. We just set them to zero for now. Let's plot $F_x(\sigma)$, slip on the horizontal axis and grip on the vertical:

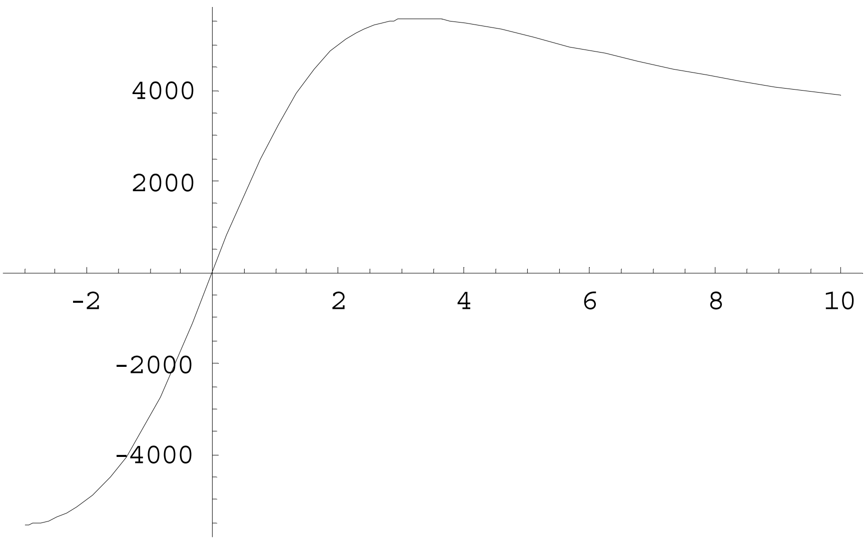


The maximum positive grip occurs, just by eyeball, around $\sigma = 0.08$. To the left of the maximum, adding more slip—more throttle—generates more grip. To the right of the maximum, adding more slip generates *less* grip. That's where we've lost traction. We can find the maximum precisely by plotting the *slope* of this curve, since the slope is zero right at the maximum:

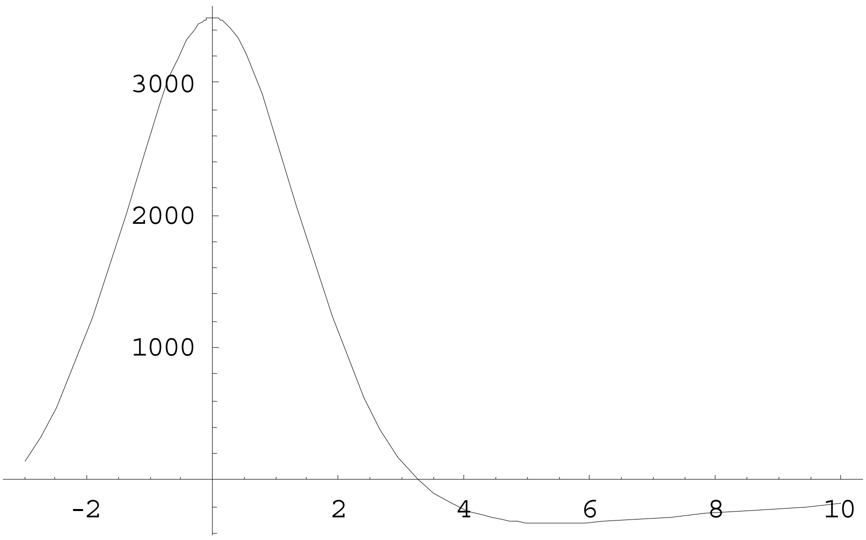


Using secret physicist methods, I've found that this curve crosses the horizontal axis—that is, goes to zero—at precisely $\sigma = 0.0796$.

This was so much fun that we'll just do it again for $F_y(\alpha)$. First, the curve proper:



Notice the same kind of stability situation as we saw before. To the left of the maximum, more slip—more steering—means more grip. To the right of the maximum, more slip means less grip. Here's the slope:



We find that the maximum of the original curve, the zero-crossing of the slope, occurs at $\alpha = 3.273^\circ$.

Once we find the maxima, we can create new, non-dimensional quantities by scaling σ and α by these values, namely $s = \sigma/\bar{\sigma}, a = \alpha/\bar{\alpha}$. These are pure numbers, so they're commensurable. They are unity when σ and α have the values of maximum traction in isolation of one another. We can then write new functions $\Phi_x(s) = F_x(\sigma)$ and $\Phi_y(a) = F_y(\alpha)$ which have their maxima at $s = 1$ and $a = 1$.

We seek a vector-valued function $\mathbf{F}^+(s,a)$ of s and a whose longitudinal x component F_x^+ expresses the longitudinal force component and whose lateral y component F_y^+ expresses the lateral force component under combination slip. Build this up from Φ_x and Φ_y so that it satisfies the following requirements:

The magnitude of $\mathbf{F}^+$, that is, $\left\|\mathbf{F}^+\right\| \equiv F^+ \equiv +\sqrt{\left(F_x^+\right)^2 + \left(F_y^+\right)^2}$, should have its maximum all the way around the traction circle, that is, whenever $+\sqrt{s^2+a^2}=1$.

The individual components should agree completely with the old magic formulae whenever there is pure longitudinal or pure lateral slip. Mathematically, this means that $F_x^+(s,0)=F_x(\sigma)=F_x(s\bar{\sigma})$ and $F_y^+(0,a)=F_y(\alpha)=F_y(a\bar{\alpha})$.

For a fixed, positive value of σ (throttle), as α (steering) increases, the input to F_x must *increase*. Say *what*? Here's the idea. Suppose you're on the limit of longitudinal grip. When steering increases, the forward grip limit must be exceeded, and a great way to model that is just to shove the input over the cliff to larger σ. We want the same behaviour the other way, namely, for a fixed value of α (steering), as σ (throttle) increases, the input to F_y increases to model the fact that at maximum steering adding throttle exceeds the limit. We model the three other cases entailing negative values of σ and α below.

Below the limits, we do not want dramatic increases in forward grip when steering increases, and vice versa. So, although we must increase the input to F_x with increasing α, we must *decrease* the output of F_x. Likewise, while we increase the input to F_y with increasing σ, we must decrease the output. This requirement is a bit of a balancing act because often there *is* an increase of steering grip with braking, as we see in the technique of trail braking. However, there is usually no increase in steering grip with increased throttle in a front-wheel-drive car, even below the limits. In the modelling of combined effects like this, it's necessary to include weight transfer with the combination grip formula. That simply means that until we have a full model of the car up and running, we won't be able to evaluate fully the quality of this combination magic grip formula.

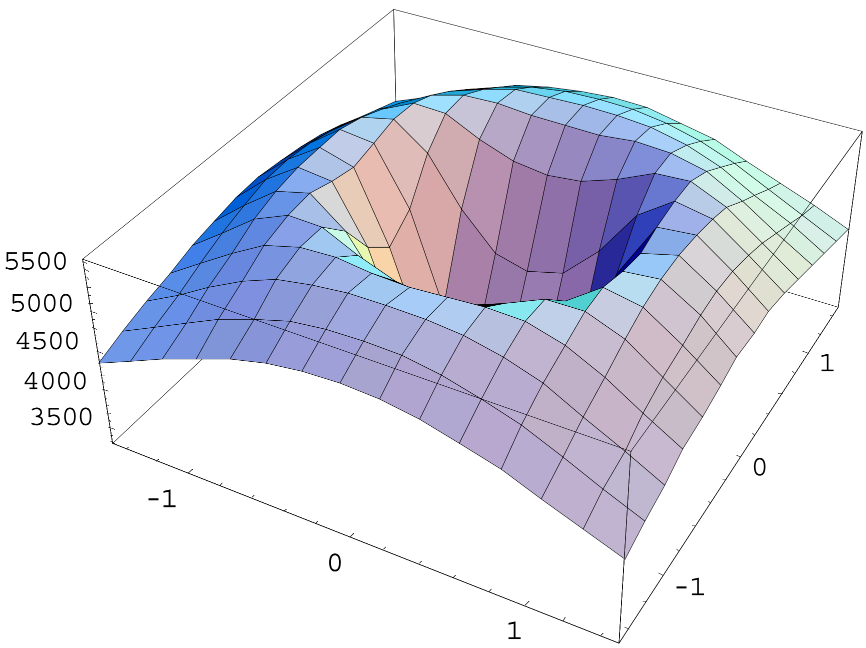
The following table fleshes out requirement 3 for the cases of braking ($\sigma<0$) or turning left ($\alpha<0$). The essential idea is that if the magnitude of either parameter increases, then the magnitudes of the inputs to the old magic formulae must increase, but honouring the algebraic signs. If a parameter is positive, it should get more positive as the magnitude of the other parameter increases. Similarly, if a parameter is negative, it should get more negative as the magnitude of the other parameter increases.

$\operatorname{sgn}(\sigma)$	$\operatorname{sgn}(\alpha)$	σ Trend	α Trend	input to F_x	input to F_y
+	+	increasing	fixed	increasing	increasing
+	+	fixed	increasing	increasing	increasing
+	–	increasing	fixed	increasing	decreasing
+	–	fixed	decreasing	increasing	decreasing
–	+	decreasing	fixed	decreasing	increasing
–	+	fixed	increasing	decreasing	increasing
–	–	decreasing	fixed	decreasing	decreasing
–	–	fixed	decreasing	decreasing	decreasing

Without further ado, here's our proposal for the combination magic grip formula:

$$\boxed{\rho = +\sqrt{s^2+a^2}\ ,\ F_x^+(s,a) = \frac{s}{\rho}\Phi_x(\rho),\ F_y^+(s,a) = \frac{a}{\rho}\Phi_y(\rho)}$$

Using ρ as the input, with the appropriate algebraic signs, satisfies requirements 1. Multiplying the outputs by the ratio of s to ρ and a to ρ magically satisfies requirements 2, 3, and 4. There is, in fact, plenty of freedom in the choice of the outer multiplier: strictly speaking, any power of the ratios would do for requirements 2 and 4, and some care will be required to get the signs right for requirement 3. Until we have a good reason to change it, we'll just go with the ratio straight up, especially since it automatically gets the signs right. We close this instalment with a plot of the magnitude $+\sqrt{\left(F_x^+\right)^2+\left(F_y^+\right)^2}$ showing the traction circle very clearly:



The stability criteria are visually obvious, here. If the current, commensurable slip values, s and a, are inside the central "cup" region, then increasing either component of slip increases grip. If they're outside, then increasing slip leads to decreasing grip and the driver is in the "deep kimchee" region of the plot.

ERRATA: The *Physics of Racing* series has been fairly error-free over the years, but I caught three small errors in part 22 whilst going over it for this instalment. The good news is that they did not affect any final results. I defined the WHEEL frame at the wheel hub but later I implied that it is centred at the contact patch (CP). In fact, the frame at the CP is the important one, and we call it TYRE from now on, avoiding the ambiguous "WHEEL". We never actually used the improperly defined WHEEL frame, so, again, final results were not affected. Also, the dimensions for a_3 in Part 22 should be N/Degree, not just N, because a_3 furnishes the dimensions for *B*, which always appears in the combination *SB*, and *S* has dimensions of degrees. Finally, the dimensions for a_6 are 1/KN, not KN.